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AN ANOMALY IN THE SIMILARITY ANALYSIS OF FILM-COOLING HEAT TRANSFER

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NOMENCLATURE

d, film-cooling slot width;
f, function;
h, heat-transfer coefficient (defined in text);
k, thermal conductivity of fluid;
L, length;
Nu, Nusselt number based on d;
p, pressure;
Pr, Prandtl number;
q_w, wall heat flux;
Re, Reynolds number based on d;
T, temperature;
u, v, velocity components;
x, y, coordinates.

Greek symbols

ζ, effectiveness, $=(T_{aw} - T_\infty)/(T_c - T_\infty)$;
θ, dimensionless temperature (defined in text).

Subscripts

a, adiabatic wall case;
aw, adiabatic wall value;
c, coolant value;
h, heat transfer case;
p, pressure;
u, v, velocity;
w, wall;
θ, temperature;
∞, mainstream value.

Superscript

$'$, dimensionless value (defined in text).

1. INTRODUCTION

FILM-COOLING, which is the protection of a surface from a hot gas by means of coolant injection adjacent the wall, has important practical applications and numerous examples are readily called to mind. In an excellent review paper on film-cooling, Goldstein [1] reports the results of many investigations, the majority of which involved empirical correlation of the data.

Of special interest in this field of study, are the adiabatic wall temperature, T_{aw}, and the associated 'effectiveness', ζ; these appear to have been foremost in the minds of the investigators. An equally important parameter is, of course, the heat-transfer coefficient, h, which in the film-cooling situation is defined as

$$h = q_w/(T_{aw} - T_w). \tag{1}$$

It is to be borne in mind, that coolant injection may significantly alter the flow field and hence the value of h. This particular problem has been studied recently by numerical methods [2] and it is evident that the local values of the heat transfer coefficient with and without film-cooling may be markedly different. As a consequence of this latter investigation, an interesting anomaly arose which was considered worthy of further enquiry. An examination of the literature indicated what variables might be expected to appear in any analysis of the problem, but a similarity study of the pertinent conservation equations revealed an unexpected result, It is with this particular part of the investigation [2] that the present technical note is concerned.

2. SIMILARITY ANALYSIS OF FILM-COOLING

Figure 1 shows the film-cooling model and Table 1 lists the appropriate boundary conditions, where

$$x' = \frac{x}{d}; \quad y' = \frac{y}{d}; \quad u' = \frac{u}{u_\infty}; \quad v' = \frac{v}{u_\infty}; \quad p' = \frac{p - p_\infty}{\rho u_\infty^2}$$

where ρ = density and

$$\theta_a' = \left[\frac{T - T_\infty}{T_c - T_\infty}\right]; \quad \theta_h' = \left[\frac{T - T_\infty}{T_w - T_\infty}\right]$$

for the adiabatic wall case and the heat transfer case respectively.

It is readily shown that the non-dimensional forms of the conservation equations for steady constant property laminar flow are

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0 \tag{2}$$

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\frac{\partial p'}{\partial x'} + \frac{1}{Re}\left[\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}\right] \tag{3}$$

(and a similar equation for the y-direction)

$$u'\frac{\partial \theta'}{\partial x'} + v'\frac{\partial \theta'}{\partial y'} = \frac{1}{RePr}\left[\frac{\partial^2 \theta'}{\partial x'^2} + \frac{\partial^2 \theta'}{\partial y'^2}\right] \tag{4}$$

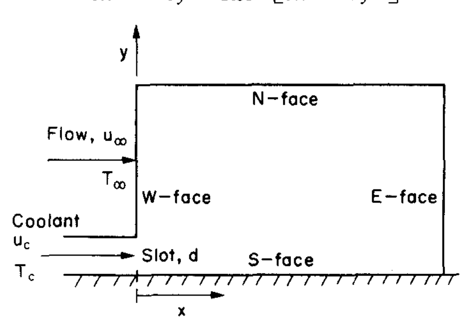

FIG. 1. Film cooling model.

Table 1. Boundary conditions

Adiabatic wall temperature case, $\theta'_a = (T - T_\infty/T_c - T_\infty)$				
Variable	N-face	S-face	E-face	W-face
u'	1	0	$f_u(y')$	$f(u'_c)$
v'	0	0	$f_v(y')$	0
p'	0	—	—	—
θ'_a	0	$\frac{\partial \theta'_a}{\partial y'} = 0$	$f_\theta(y')$	$y' \geq 1$, 0; $y' < 1$, 1
Heat transfer case, $\theta'_h = (T - T_\infty/T_w - T_\infty)$				
Variable	N-face	S-face	E-face	W-face
u'	1	0	$f_u(y')$	$f(u'_c)$
v'	0	0	$f_v(y')$	0
p'	0	—	—	—
θ'_h	0	1	$f_\theta(y')$	$y' \geq 1$, 0; $y' < 1$, θ'_{hc}

where the Reynolds number, $Re = (u_\infty d/\nu)$ and ν = kinematic viscosity.

Now the examination of these four equations indicates that they are second order in both x' and y' for the velocities and the temperature, and first order in x' and y' for the pressure. Accordingly, four boundary conditions are required for each of u', v' and θ' and one boundary condition for p' in x' and also in y' if a solution is to be effected. The boundary conditions are listed for the two relevant cases in Table 1.

If a solution is possible, then the dependent variables u', v', p' and θ' may be expressed in terms of the independent variables x' and y' and (according to Eckert and Drake [3]) 'all constant parameters contained in the differential equations and in the boundary conditions'.

Therefore, for the adiabatic wall case

$$u' = f_u(x', y', Re, Pr, u'_c) \tag{5}$$

$$v' = f_v(x', y', Re, Pr, u'_c) \tag{6}$$

$$p' = f_p(x', y', Re, Pr, u'_c) \tag{7}$$

$$\theta'_a = f_\theta(x', y', Re, Pr, u'_c) \tag{8}$$

it being remembered that θ'_a is defined as shown in Table 1.

The adiabatic wall distribution is obtained from equation (8) with $y' = 0$ as

$$\theta'_{a(y'=0)} = f_\theta(x', Re, Pr, u'_c) \tag{9}$$

which is the result anticipated and supported by experimental evidence [1]. That is, for a given fluid, the local adiabatic wall temperature is dependent on the injection velocity ratio, u'_c, in addition to the main flow velocity.

Referring now to the case when heat is transferred at the wall, the following results are obtained

$$u' = f_u(x', y', Re, Pr, u'_c, \theta'_{hc}) \tag{10}$$

$$v' = f_v(x', y', Re, Pr, u'_c, \theta'_{hc}) \tag{11}$$

$$p' = f_p(x', y', Re, Pr, u'_c, \theta'_{hc}) \tag{12}$$

$$\theta'_h = f_\theta(x', y', Re, Pr, u'_c, \theta'_{hc}). \tag{13}$$

It is important to note at this point:

(1) that θ' is redefined as indicated in Table 1, and;

(2) the appearance of θ'_{hc} in the functions in addition to the injection velocity ratio.

The analysis may be developed even further to obtain an expression for the heat transfer coefficient, h, or its dimensionless counterpart, the Nusselt number, Nu, and of course the general results are valid for turbulent flow.

Since,

$$h = \frac{q_w}{T_{aw} - T_w} = -k\frac{\partial T}{\partial y}\bigg|_{y=0} \bigg/ T_{aw} - T_w \tag{14}$$

Substitutions for T using both θ'_a and θ'_h defined previously may be made to produce

$$Nu = -\frac{\partial \theta'_h}{\partial y'}\bigg|_{y'=0} \left\{1 \bigg/ \left(\frac{T_{aw} - T_\infty}{T_w - T_\infty}\right) - 1\right\} \tag{15}$$

$$Nu = -\frac{\partial \theta'_h}{\partial y'}\bigg|_{y'=0} \left\{1 \bigg/ \left(\frac{T_{aw} - T_\infty}{T_c - T_\infty}\right)\left(\frac{T_c - T_\infty}{T_w - T_\infty}\right) - 1\right\} \tag{16}$$

$$Nu = \frac{f'_\theta(x', Re, Pr, u'_c, \theta'_{hc})}{\theta'_{a(y'=0)}\theta'_{hc} - 1} \tag{17}$$

$$Nu = \frac{f_\theta(x', Re, Pr, u'_c, \theta'_{hc})}{f_\theta(x', Re, Pr, u'_c)\theta'_{hc} - 1} \tag{18}$$

$$Nu = f_\theta(x', Re, Pr, u'_c, \theta'_{hc}). \tag{19}$$

Here then is the anomaly referred to earlier. Unlike other forced convection correlations, the temperature level (of the injected fluid in this case) appears as a variable in the problem which is not in accordance with experimental evidence and the numerical prediction of constant property turbulence film-cooling, [2]. A possible explanation of this anomaly is as follows.

Consider unidirectional steady state heat conduction for which the second-order differential equation is

$$\frac{d^2 T}{dx^2} = 0. \tag{20}$$

If the boundary conditions are $T = T_1$, at $x = 0$, and $T = T_2$ at $x = L$, then the solution for the temperature, T, is easily shown to be

$$T = T_1\left(1 - \frac{x}{L}\right) + T_2\left(\frac{x}{L}\right). \tag{21}$$

In other words, the temperature is the sum of two terms each of which is a boundary value multiplied by a function which is independent of the boundary values. Each of these functions is a degree less than the order of the original differential equation, and each term satisfies the original differential equation.

In the present problem on film-cooling, the four boundary conditions for the temperature, θ'_h, include the coolant temperature θ'_{hc} (the anomalous parameter) in the final expression for the heat transfer coefficient. On the basis of what has been observed concerning the form of the solution for unidirectional heat conduction, it is possible that the boundary value may be removed from the function in equation (19) leaving a function which is independent of the boundary temperatures. Accordingly, the heat-transfer coefficient may then be

$$Nu = f_\theta(x', Re, Pr, u'_c) \tag{22}$$

which is consistent with the accepted form of convective heat transfer correlations with temperature independent fluid properties. That is, for a given fluid, the local heat transfer rate in convective heat transfer is determined by the geometry and flow characteristics only, when the properties of the fluid are independent of temperature.

3. CONCLUSION

A rigorous treatment of film-cooling heat transfer by similarity analysis results in an anomalous result concerning the parameters on which the local heat transfer coefficient depends. By examining the form of the solution for a simple heat conduction problem involving boundary conditions, a suggestion has been made as to how the irregularity may be removed from the film-cooling problem.

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